ſunt, comparationes eo magis ſunt notatu dignæ, quo minus via ad eas comprobandas patere videatur. Sic primum hujus generis theorema, ad quod jam pridem fui deductus, ſimplicitate ſe commendabat, quo inveni eſſe productum harum duarum formularum integralium $\int \frac{dz}{\sqrt{(1-z^4)}}$ & $\int \frac{z^2 dz}{\sqrt{(1-z^4)}}$ quarum altera arcum, altera ordinatam in curva elaſtica exprimit, caſu quo $z = 1$, æquale areæ circuli cujus diameter ſit $= 1$.

Num. III.

Ejusdem

De Inventione Integralium ſi poſt integrationem variabili quantitati determinatus valor tribuatur.

Lemma 1.

1. Invenire ſummam ſeriei recurrentis $A + B + C + D + \ldots\ldots + P$ in qua quilibet terminus ex duobus præcedentibus ita formatur ut ſit $C = mB + nA$, $D = mC + nB$; &c. Tab II.

Solutio.

Quo ſolutio latius pateat, multiplicemus ſingulos terminos per terminos progreſſionis geometricæ, ut habeamus hanc ſeriem:

S Ax

$$\overset{1}{A}x^{\alpha} + \overset{2}{B}x^{\alpha+\beta} + \overset{3}{C}x^{\alpha+2\beta} + \overset{4}{D}x^{\alpha+3\beta} + \ldots + \overset{p}{P}x^{\alpha+(p-1)\beta}$$

Ponamus hujus ſerici ſummam $= S$, ut ſit

$$S = Ax^{\alpha} + Bx^{\alpha+\beta} + Cx^{\alpha+2\beta} + \ldots\ldots + Px^{\alpha+(p-1)\beta}$$

Hinc lege progreſſionis in computum ducta erit

$$mSx^{\beta} = mAx^{\alpha+\beta} + mBx^{\alpha+2\beta} + \ldots\ldots + mOx^{\alpha+(p-1)\beta} + mPx^{\alpha+p\beta}$$

$$nSx^{2\beta} = nAx^{\alpha+2\beta} + \ldots\ldots + nNx^{\alpha+(p-1)\beta}$$
$$+ nOx^{\alpha+p\beta} + nPx^{\alpha+(p+1)\beta}$$

ſubtrahantur hæ duæ ſeries conjunctim a ſuperiori atque ob $C = mB + nA$, $D = mC + nB$; $\ldots\ldots$ $P = mO + nN$ habebitur: $S(1 - mx^{\beta} - nx^{2\beta}) = Ax^{\alpha} + Bx^{\alpha+\beta} - mAx^{\alpha+\beta} - mPx^{\alpha+p\beta} - nOx^{\alpha+p\beta} - nPx^{\alpha+(p+1)\beta}$. Sit in ſerie propoſita $A + B + C + D + \ldots\ldots + P$ terminus ultimum P ſequens $= Q$ erit $Q = mP + nO$, quo introducto fiet

$$S = \frac{Ax^{\alpha} + Bx^{\alpha+\beta} - mAx^{\alpha+\beta} - Qx^{\alpha+p\beta} - nPx^{\alpha+(p+1)\beta}}{1 - mx^{\beta} - nx^{2\beta}}$$

Vel ſi in ſerie $A + B + C + \ldots\ldots + P$ vocetur terminus primum A antecedens $= \Delta$, propter $B = mA + n\Delta$ fiet ſumma quæſita

$$S = \frac{Ax^{\alpha} + n\Delta x^{\alpha+\beta} - Qx^{\alpha+p\beta} - nPx^{\alpha+(p+1)\beta}}{1 - mx^{\beta} - nx^{2\beta}}$$

Facto

Facto jam $x = 1$ erit seriei propositæ

$A + B + C + \ldots\ldots + P$ summa $=$

$\frac{A + n\Delta - Q - nP}{1 - m - n}$. Q. E. J.

Lemma 2.

2. Existente $A + B + C + D + \ldots\ldots + P$ serie recurrente, in qua sit $C = mB + nA$; $D = mC + nB$, &c. invenire summam hujus seriei:

$\alpha A + (\alpha + \beta) B + (\alpha + 2\beta) C + \ldots\ldots + (\alpha + (p-1)\beta) P$.

Solutio.

Consideremus seriem latius patentem hanc.

$$S = Ax^{\alpha} + Bx^{\alpha+\beta} + Cx^{\alpha+2\beta} + \ldots\ldots Px^{\alpha+(p-1)\beta}$$

cujus summam ante invenimus esse:

$$S = \frac{Ax^{\alpha} + n\Delta x^{\alpha+\beta} - Qx^{\alpha+p\beta} - nPx^{\alpha+(p+1)\beta}}{1 - mx^{\alpha} - nx^{2\beta}}$$

denotante Δ terminum primum A antecedentem, ac Q terminum ultimum P sequentem in serie

$A + B + C + D + \ldots\ldots + P$.

Quodsi jam posito x variabili differentietur series, cujus summam posuimus $= S$ erit

$$\frac{dS}{dx} = \alpha Ax^{\alpha-1} + (\alpha+\beta) Bx^{\alpha+\beta-1} + \ldots\ldots + (\alpha + (p-1)\beta) Px^{\alpha+(p-1)\beta-1}$$

ut

at ex valore summæ S ante invento erit

$$\frac{dS}{dx} = \frac{\begin{array}{l} \alpha A x^{\alpha-1} + m(\beta-\alpha)A x^{\alpha+\beta-1} + n(2\beta-\alpha)A x^{\alpha+2\beta-1} \\ \quad + n(\beta+\alpha)\Delta \quad - mn\ \alpha\ \Delta \quad nn(\beta-\alpha)\Delta x^{\alpha+3\beta-1} \\ -(\alpha+p\beta)Q x^{\alpha+p\beta-1} + m(\alpha+(p-1)\beta)Q x^{\alpha+(p+1)\beta-1} + n(\alpha+(p-2)\beta)Q x^{\alpha+(p+2)\beta-1} \\ \quad - n(\alpha+(p+1)\beta)P \quad + mn(\alpha+p\beta)P \\ \quad + nn(\alpha+(p-1)\beta)P x^{\alpha+(p+3)\beta-1} \end{array}}{(1-mx^{\beta}-nx^{2\beta})^2}$$

Ponatur jam $x = 1$ eritque seriei propositæ

$\alpha A + (\alpha+\beta) B + (\alpha+2\beta) C + \ldots\ldots + (\alpha+(p-1)\beta) P$

$$\text{summa} = \frac{\begin{array}{l} +(1-m-n)\alpha A + (m+2n)\beta A + n(1-m-n)\alpha\Delta + n(1+n)\beta\Delta \\ -(1-m-n)\alpha Q - p(1-m-n)\beta Q - (m+2n)\beta Q - n(1-m-n)\alpha P \\ -np(1-m-n)\beta P - n(1+n)\beta P \end{array}}{(1-m-n)^2}$$

Vel hæc summa est $\dfrac{\alpha A + n\alpha\Delta - (\alpha+p\beta)Q - n(\alpha+p\beta)P}{1-m-n}$

$$+ \frac{(m+2n)\beta A + n(1+n)\beta\Delta - (m+2n)\beta Q - n(1+n)\beta P}{(1-m-n)^2}$$

Q. E. J.

Coroll. 1.

3. Summa ergo hujus seriei:

$A + 2B + 3C + 4D + \ldots\ldots pP$

erit $= \dfrac{A+n\Delta-(1+p)Q-n(1+p)P}{1-m-n} + \dfrac{(m+2n)(A-Q)+n(1+n)(\Delta-P)}{(1-m-n)^2}$

existente $A+B+C+D+\ldots\ldots+P$ serie recurrente cujus indices sint $m+n$.

Coroll. 2.

Coroll. 2.

4. Simili modo hujus ſeriei:

$$A + 3B + 5C + 7D + \ldots\ldots (2p-1)P$$

ſumma erit $= \dfrac{A + n\Delta - (2p+1)Q - n(2p+1)P}{1-m-n}$

$$+ \frac{2(m+2n)(A-Q) + 2n(1+n)(\Delta - P)}{(1-m-n)^2}.$$

Problema. 1.

5. Invenire ſummam ſinuum quotcunque angulorum in progreſſione arithmetica progredientium.

Solutio.

Teneant anguli, quorum ſinuum ſumma quæritur, hanc progreſſionem:

$$\overset{1}{s};\ \overset{2}{s+u};\ \overset{3}{s+2u};\ \overset{4}{s+3u};\ \ldots\ldots\ldots\ \overset{p}{s+(p-1)u}$$ erit ergo ſeries ſinuum ſummanda hæc $\sin A s + \sin A(s+u) + \sin A(s+2u) + \ldots + \sin A(s+(p-1)u)$ Eſt vero hæc progreſſio ſeries recurrens, cujus indices ſunt $2\cos A u$, -1 ſumta unitate pro ſinu toto; unde erit $m = \cos A u$ & $n = -1$, facta ad lemma 1 applicatione. Porro erit $A = \sin A s$; $P = \sin A(s+(p-1)u)$; $Q = \sin A(s+pu)$ & $\Delta = \sin A(s-u)$. Hinc erit ſumma ſeriei ſinuum propoſitæ $= \dfrac{\sin A s - \sin A(s-u) - \sin A(s+pu) + \sin A(s+(p-1)u)}{2 - 2\cos A u}$

$= \sin A s + \sin A(s+u) + \sin A(s+2u) + \ldots\ldots + \sin A(s+(p-1)u)$

Q. E. J.

Coroll. 1.

6. Quoniam eſt $\sin A(s-u) = \sin A s.\ \cos A u - \cos A s.\ \sin A u$

 erit

erit $\frac{\sin A s - \sin A(s-u)}{2 - 2\cos A u} = \frac{\sin A s}{2} + \frac{\cos A s . \sin A u}{2(1-\cos A u)} =$ $\frac{\sin A s}{2} + \frac{\cos A s}{2 \tan \frac{1}{2} u}$, ob $\frac{\sin A u}{1-\cos A u} = \frac{1}{\tan A \frac{1}{2} u}$; simili modo est $\sin A(s+(p-1)u) = \sin A(s+pu)\cos A u - \cos A(s+pu)\sin A u$ hincque $-\frac{\sin A(s+pu) + \sin A(s+(p-1)u)}{2(1-\cos A u)} =$ $-\frac{\sin A(s+pu)}{2} - \frac{\cos A(s+pu)}{2 \tan A \frac{1}{2} u}$; unde erit seriei propositæ summa $= \frac{\sin A s - \sin A(s+pu)}{2} + \frac{\cos A s - \cos A(s+pu)}{2 \tan A \frac{1}{2} u}$.

Coroll. 2.

7. Quia porro est $\tan A \frac{1}{2} u = \frac{\sin A \frac{1}{2} u}{\cos A \frac{1}{2} u}$; erit seriei propositæ summa $= \frac{\cos A(s - \frac{1}{2}u) - \cos A(s + pu - \frac{1}{2}u)}{2 \sin A \frac{1}{2} u}$

In serie igitur arcuum a primo s subtrahatur dimidia differentia ($\frac{1}{2} u$) eademque ad ultimum arcum addatur, arcuumque resultantium cosinus hujus subtrahatur a cosinu illius, ac differentia per duplum sinum dimidiæ differentiæ divisa, dabit summam omnium sinuum arcuum illorum arithmeticam progressionem constituentium.

Coroll. 3.

8. Si semicirculus cujus radius $= 1$, dividatur in partes quotcunque æquales numero n; erit posita semicircumferentia $= \pi$, differentia $= \frac{\pi}{n}$. Quod si jam ex singulis divisionis

onis punctis ſinus ad diametrum ducantur, erit ob $s = \frac{\pi}{n}$; $u = \frac{\pi}{n}$; & $s + (p-1)\,u = \pi$, ſumma omnium horum ſinuum $= \frac{\cos A.\ \frac{\pi}{2n} - \cos A\left(\pi + \frac{\pi}{2n}\right)}{2 \sin A\ \frac{\pi}{2n}} = \cot A. \frac{\pi}{2n}$

Coroll. 4.

9. Quod ſi igitur ſemicirculus A D G in partes quotcunque æquales AB, BC, CD, &c. dividatur, atque ex ſingulis diviſionum punctis B, C, D, E &c. ad diametrum AG demittantur normales Bb, Cc, Dd, Ee, & Ff, ſumma harum rectarum junctim ſumtarum æquabitur cotangenti ſemiſſis unius partis, ſeu biſecta prima parte AB in M, ductaque hujus ſemiſſis AM tangente AT erit AT ad radium uti radius ad ſummam omnium ſinuum Bb + Cc + Dd + Ee + Ff, quod eſt Theorema Vietæ. *Fig. I.*

Coroll. 5.

10. Simili modo ſi arcus circuli quicunque BG ſecetur in partes quotcunque æquales BC, CD, DE, &c. atque ex ſingulis iſtis punctis ad diametrum quampiam pro lubitu ductam AOS demittantur perpendicula, Bb, Cc, Dd . . . . . Gg, hæc perpendicula erunt ſinus arcuum AB, AC, AD, ... AG in arithmetica progreſſione progredientium, exiſtente AB$=s$, BC$=u$, & AG $= s + (p-1)\,u$. Quare ſi utrinque ad arcum diviſum BG addantur partes BM$=$GN $= \frac{1}{2}$ BC, hincque demittantur perpendicula Mm & Nn erit Om $= \cos A\ (s - \frac{1}{2}\,u)$ & On $= - \cos A\,(s + (p - \frac{1}{2})\,u)$. *Fig. II.*

Chorda

Chorda autem BC erit $= 2 \cdot \sin A \frac{1}{2} u$. Ex his ergo reperitur omnium sinuum $Bb + Cc + Dd + Ee + Ff + Gg$ summa $= \frac{Om + On}{2, BC} = \frac{mn}{2BC}$, posito radio $OA = 1$. Hinc si super basi mn construatur triangulum isosceles simile triangulo BOC, cordam BC pro basi & centrum O pro vertice habenti, tum unum crus istius trianguli æquale erit summæ sinuum $Bb + Cc + Dd + Ee + Ff + Gg$.

Problema 2.

Fig. 3. 11. Diviso semicirculo in partes quotcunque æquales AB, BC, CD, &c. demissisque sinibus Bb, Cc, Dd, Ee, &c. compleantur parallelogramma rectangula $b\alpha$, $c\beta$, $d\gamma$, $e\delta$, $f\epsilon$, $g\zeta$, $h\eta$, $i\theta$, $k\iota$; quorum omnium junctim sumtorum determinari summam oporteat.

Solutio.

Posito radio $AO = 1$, & semicircumferentia $= \pi$ sit ea divisa in partes æquales, numero n, erit unaquæque partium $AB = BC = CD = \&c. = \frac{\pi}{n}$; hincque $Bb = \sin A \frac{\pi}{n}$; $Cc = \sin A \frac{2\pi}{n}$; $Dd = \sin A \frac{3\pi}{n}$; &c. usque ad ultimum divisionis punctum k, pro quo sinus erit $= \sin A \frac{n\pi}{n} = o$. Jam parallelogammorum bases erunt ut sequitur $Ab = 1 - \cos A \frac{\pi}{n}$, $bc = \cos A \frac{\pi}{n} - \cos A \frac{2\pi}{n}$; $cd =$

cos

$\cos A\frac{2\pi}{n} - \cos A\frac{3\pi}{n}$, $de = \cos A\frac{3\pi}{n} - \cos A\frac{4\pi}{n}$; & ultima basis $iK = \cos A\frac{(n-1)\pi}{n} - \cos A\frac{n\pi}{n}$, cui respondet altitudo $= 0$. Cum porro sit generaliter $\sin A\,\phi . \cos A\,\psi = \frac{\sin A\,(\phi+\psi) + \sin A\,(\phi-\psi)}{2}$, areæ parallelogrammmorum nostrorum ita se habebunt.

$$b\alpha = \sin A\frac{\pi}{n}\left(1-\cos A\frac{\pi}{n}\right) = \tfrac{1}{2}\sin A\frac{\pi}{n} + \tfrac{1}{2}\sin A\frac{\pi}{n} - \tfrac{1}{2}\sin A\frac{2\pi}{n}$$

$$c\beta = \sin A\frac{2\pi}{n}\left(\cos A\frac{\pi}{n} - \cos A\frac{2\pi}{n}\right) = \tfrac{1}{2}\sin A\frac{\pi}{n} + \tfrac{1}{2}\sin A\frac{3\pi}{n} - \tfrac{1}{2}\sin A\frac{4\pi}{n}$$

$$d\gamma = \sin A\frac{3\pi}{n}\left(\cos A\frac{2\pi}{n} - \cos A\frac{3\pi}{n}\right) = \tfrac{1}{2}\sin A\frac{\pi}{n} + \tfrac{1}{2}\sin A\frac{5\pi}{n} - \tfrac{1}{2}\sin A\frac{6\pi}{n}$$

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$$K2 = \sin A\frac{n\pi}{n}\left(\cos A\frac{(n-1)\pi}{n} - \cos A\frac{n\pi}{n}\right) = \tfrac{1}{2}\sin A\frac{\pi}{n} + \tfrac{1}{2}\sin A\frac{(2n-1)\pi}{n} - \tfrac{1}{2}\sin A\frac{2n\pi}{n}$$

Tres igitur series habemus, quarum summas investigare debemus, ac primæ quidem cujus omnes termini sunt $= \frac{1}{2}\sin A\frac{\pi}{n}$ summa erit $= \frac{n}{2}\sin A\frac{\pi}{n}$. Secunda series duplicata est

sin

$\sin A\,\frac{\pi}{n} + \sin A\,\frac{3\pi}{n} + \sin A\,\frac{5\pi}{n} + \ldots\ldots + \sin A\,\frac{(2n-1)\pi}{n}$

quæ ad propoſitionem præcedentem accommodata dat $s = \frac{\pi}{n}$; $u = \frac{2\pi}{n}$; $s + (p-1)\,u = \frac{\pi}{n} + \frac{2(p-1)\pi}{n} = \frac{(2n-1)\pi}{n}$.

Ejus ergo ſumma erit $= \dfrac{\cos A\left(\frac{\pi}{n} - \frac{\pi}{n}\right) - \cos A\,\frac{2n\pi}{n}}{2\sin A\,\frac{\pi}{n}} = 0.$

Tertia ſeries bis ſumta eſt:

$\sin A\,\frac{2\pi}{n} + \sin A\,\frac{4\pi}{n} + \sin A\,\frac{6\pi}{n} + \ldots\ldots + \sin A\,\frac{2n\pi}{n}$

quæ ergo dat $s = \frac{2\pi}{n}$; $u = \frac{2\pi}{n}$; ex quo ipſius ſumma erit $= \dfrac{\cos A\,\frac{\pi}{n} - \cos A\,\frac{(2n+1)\pi}{n}}{2\sin A\,\frac{\pi}{n}} = \dfrac{\cos A\,\frac{\pi}{n} - \cos A\,\frac{\pi}{n}}{2\sin A\,\frac{\pi}{n}} = 0$

Cum igitur ſecundæ & tertiæ ſeriei ſummæ evaneſcant, erit ſumma omnium rectangulorum quæſita:

$b\,\alpha + c\,\beta + d\,\gamma + e\,\delta + \ldots + K\iota = \frac{n}{2}\sin A\,\frac{\pi}{n}$ Q. E. J.

Corollarium.

12. Si ergo diametro AK ducatur parallela $m\,o$, quæ tangentem $A\alpha$ biſecet in m & excentro O erigatur perpendicularis Oo, erit $Oo = Am = \frac{1}{2}\,Bb = \frac{1}{2}\sin A\,\frac{\pi}{n}$; hincque area rectanguli OomA ob radium $AO = 1$ erit $= \frac{1}{2}\sin A\,\frac{\pi}{n}$; hoc igitur rectangulum toties ſumtum, quot ſunt diviſionis puncta, ſeu quot numerus n continet unitates dabit

bit summam omnium rectangulorum $b\alpha + c\beta + d\gamma +$ &c.

Problema. 3.

13. Si arcus circuli quicunque BH dividatur in partes quotcunque æquales BC, CD, DE, &c. atque ex singulis divisionis punctis ad diametrum pro lubitu ductam demittantur perpendicula Bb, Cc, Dd, &c. ac præterea ex his parallelogramma compleantur $c\beta$; $d\gamma$; $e\delta$; $f\epsilon$; $g\zeta$, $h\eta$, invenire aream omnium horum parallelogrammorum junctim sumtorum.

Solutio.

Sit arcus divisus BH $= q$; numerus divisionum $= n$, ita ut qualibet pars BC $=$ CD $=$ DE &c. futura sit $= \frac{q}{n}$. Sit præterea arcus AB $= a$, erit Bb $=$ sin Aa; Cc $=$ sin A $\left(a + \frac{q}{n}\right)$; Dd $=$ sin A $\left(a + \frac{2q}{n}\right)$, &c. ultima vero Hh $=$ sin A $\left(a + \frac{nq}{n}\right) =$ sin A $(a+q)$. Ex his rectangula proposita ita se habebunt: *Fig. IV.*

$$c\beta = \sin A\left(a+\frac{q}{n}\right)\left(\cos A a - \cos A\left(a+\frac{q}{n}\right)\right) = \tfrac{1}{2}\sin A\,\frac{q}{n} + \tfrac{1}{2}\sin A\left(2a+\frac{q}{n}\right) - \tfrac{1}{2}\sin A\left(2a+\frac{2q}{n}\right)$$

$$d\gamma = \sin A\left(a+\frac{2q}{n}\right)\left(\cos A\left(a+\frac{q}{n}\right) - \cos A\left(a+\frac{2q}{n}\right)\right) = \tfrac{1}{2}\sin A\,\frac{q}{n} + \tfrac{1}{2}\sin A\left(2a+\frac{3q}{n}\right) - \tfrac{1}{2}\sin A\left(2a+\frac{4q}{n}\right)$$

$$e\delta = \sin A\left(a+\frac{3q}{n}\right)\left(\cos A\left(a+\frac{2q}{n}\right) - \cos A\left(a+\frac{3q}{n}\right)\right) = \tfrac{1}{2}\sin A\,\frac{q}{n} + \tfrac{1}{2}\sin A\left(2a+\frac{5q}{n}\right) - \tfrac{1}{2}\sin A\left(2a+\frac{6q}{n}\right)$$

$h\eta =$

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$$bn = \sin A(a+q)\left(\cos A a+\frac{(n-1)q}{n}\right)-\cos A(a+q)\Big) = \tfrac{1}{2}\sin A\frac{q}{n} +$$
$$\tfrac{1}{2}\sin A\left(2a+\frac{(2n-1)q}{n}\right)-\tfrac{1}{2}\sin A(2a+2q)$$

Iterum igitur tres ſeries ſummari oportet, quarum primæ ſummam patet eſſe $= \frac{n}{2}\sin A\frac{q}{n}$. Secundæ ad hanc addendæ ſumma per propoſitionem 1, eſt $= \dfrac{\cos A 2a-\cos A(2a+2q)}{4\sin A\frac{q}{n}}$; tertiæ ſubtrahendæ ſumma eſt $= \dfrac{\cos A\left(2a+\frac{q}{n}\right)-\cos A\left(2a+2q+\frac{q}{n}\right)}{4\sin A\frac{q}{n}}$. Omnium ergo rectangulorum propoſitorum ſumma erit $=$

$$\frac{n}{2}\sin A\frac{q}{n}+\frac{\cos A 2a-\cos A\left(2a+\frac{q}{n}\right)-\cos A(2a+2q)+\cos A\left(2a+2q+\frac{q}{n}\right)}{4\sin A\frac{q}{n}}$$

$$= \frac{n}{2}\sin A\frac{q}{n}+\frac{\sin A\frac{q}{2n}\left(\sin A\left(2a+\frac{q}{2n}\right)-\sin A\left(a+2q+\frac{q}{2n}\right)\right)}{2\sin A\frac{q}{n}}$$

$$= \frac{n}{2}\sin A\frac{q}{n}+\frac{\sin A q\left(\sin A(2a+q)-\sin A\left(2a+q+\frac{q}{n}\right)\right)}{2\sin A\frac{q}{n}}$$

quæ

quæ reductiones eo nituntur fundamento, quo differentia cosinuum duorum angulorum æqualis est duplo producto ex sinu semisummæ in sinum semidifferentiæ eorundem angulorum. Q. E. J.

Coroll. 1.

14. Si diameter AS ab utroque arcus divisi BH termino æqualiter distet, ut sit AB = SH = a, erit $2a + q =$ semicircumferentiæ π; hincque sin A $(2a+q) = 0$ & sin A$(2a+q+\frac{q}{n}) = -$ sin A $\frac{q}{n}$. Hoc ergo casu summa omnium rectangulorum erit $= \frac{n}{2}$ sin A $\frac{q}{n} + \frac{1}{2}$ sin Aq.

Coroll. 2

15. Quoniam est sin A $\frac{q}{n} = 2$ sin A $\frac{q}{2n}$. cos A $\frac{q}{2n}$ erit ex secunda expressione summa omnium rectangulorum =

$$\frac{n}{2}\text{ sin A }\frac{q}{n} + \frac{\text{sin A }(2a+\frac{q}{2n}) - \text{sin A }(2a+2q+\frac{q}{2n})}{4\text{ cos A }\frac{q}{2n}}$$

Coroll. 3.

16. Si ponatur alterum arcus divisi complementum SH=b erit $a + b + q = \pi$, & $a = \pi - b - q$, qui valor in postremo sinu substitutus dabit summam rectangulorum quæsitam

$$= \frac{n}{2}\text{ sin A }\frac{q}{n} + \frac{\text{sin A}(2a+\frac{q}{2n}) + \text{sin A }(2b-\frac{q}{2n})}{4\text{ cos A }\frac{q}{2n}}$$

Coroll. 4.

17. Hæc expressio summæ quæsitæ reduci potest ad hanc

 for-

formam: $\frac{n}{2}$ fin A $\frac{q}{n}$ + $\frac{1}{4}$ fin A $2a$ + $\frac{1}{4}$ fin A $2b$ + $\frac{1}{4}$ tang A$\frac{q}{2n}$ (cof A $2a$ — cof A $2b$.) Hæcque ultimus transmutatur in hanc: $\frac{n}{2}$ fin A $\frac{q}{n}$ + $\frac{1}{2}$ fin A$(a+b)$ cof A$(a-b)$ — $\frac{1}{2}$ fin A$(a+b)$ fin A$(a-b)$ tang A$\frac{q}{2n}$ = $\frac{n}{2}$ fin A$\frac{q}{n}$ + $\frac{\text{fin A}(a+b)}{2\,\text{cof A}\frac{q}{2n}}$ cof A$\left(a-b+\frac{q}{2n}\right)$ = $\frac{n}{2}$ fin A $\frac{q}{n}$ + $\frac{\text{fin A}\,q.\,\text{cof A}\left(a-b+\frac{q}{2n}\right)}{2\,\text{cof A}\frac{q}{2n}}$

Problema 4.

18. Invenire fummam hujus feriei cofinuum: cof A s + cof A $(s+u)$ + ... + cof A $(s+(p-1)u)$ quorum anguli s; $s+u$; $s+2u$; .... $s+(p-1)u$ progreffionem arithmeticam conftituunt.

Solutio.

Seriei hujus cofinuum pariter ac finuum fumma ope lemmatis primi inveniri poteft, cum cofinus angulorum in arithmetica progreffione progredientium conftituant feriem recurrentem, cujus indices funt 2 cof A u, —1. erit ergo A = cof A s; Δ = cof A $(s-u)$; P = cof A $(s+(p-1)u)$; Q = cof A $(s+pu)$; & m = 2 cof A u; n = -1. ex quibus feriei propofitæ fumma erit = $\frac{\text{cof A}s - \text{cof A}(s-u) - \text{cof A}(s+pu) + \text{cof A}(s+(p-1)u).}{2 - 2\,\text{cof A}\,u.}$

Cum autem fit cof A $(s-u)$ = cof A s cof Au + fin As. fin Au atque cof $A(s+pu-u)$ = cof $A(s+pu)$ cof Au + fin $A(s+pu)$ fin Au, erit fumma = $\frac{1}{2}$ cof As — $\frac{\text{fin } A\, s}{2\,\text{tang}\,\frac{1}{2}u}$ — $\frac{1}{2}$ cof A $(s+pu)$

$$(s+pu)+\frac{\sin A(s+pu)}{2\,\mathrm{tang}\,\frac{1}{2}u}=\frac{-\sin A(s-\frac{1}{2}u)+\sin A(s+(p-\frac{1}{2})u).}{2\sin A\frac{1}{2}u}$$

Q. E. J.

Scholion.

19. Eadem cosinuum summa ex inventa sinuum summa per differentiationem facile inveniri potest. Cum enim sit; $\sin As+\sin A(s+u)+\sin A(s+2u)+\ldots\ldots\ldots+\sin A(s+(p-1)u)=\frac{\cos A(s-\frac{1}{2}u)-\cos A(s+(p-\frac{1}{2})u)}{2\sin A\frac{1}{2}u}$

differentietur hæc æquatio posito s variabili & u constante, ac facta divisione utrinque per ds fiet:

$$\cos As+\cos A(s+u)+\cos A(s+2u)+\ldots+\cos A(s+(p-1)u)=\frac{-\sin(s-\frac{1}{2}u)+\sin A(s+(p-\frac{1}{2})u).}{2\sin A\frac{1}{2}u}$$

Corollarium 1.

20. Proposita ergo serie cosinuum, quorum anguli in progressione arithmetica progrediuntur, a primo angulo subtrahatur semidifferentia progressionis, hæcque eadem semidifferentia ad angulum ultimum addatur. Tum sinus illius anguli subtrahatur a sinu hujus & differentia per duplum sinum semidifferentiæ divisa dabit summam omnium cosinuum.

Coroll. 2.

21. Si angulus primus s evanescat, & ultimus $s+(p-1)u$ fiat rectus, erit $-\sin A(s-\frac{1}{2})=\sin A\frac{1}{2}u$ & $\sin A(s+(p-\frac{1}{2})u)=\cos A\frac{1}{2}u$, ex quo hujus seriei cosinuum summa erit $=\frac{1}{2}+\frac{1}{2}\cot A\frac{1}{2}u$.

Pro-

Problema. 5.

22. Invenire ſummam hujus ſeriei ſinuum:
$\alpha \sin As + (\alpha+\beta)\sin A(s+u) + (\alpha+2\beta)\sin A(s+2u) + (\alpha+3\beta)\sin A(s+3u) + \ldots\ldots + (\alpha+(p-1)\beta)\sin A(s+(p-1)u)$ quorum coefficientes progreſſionem arithmeticam conſtituunt; anguli autem ipsi pariter in arithmetica progreſſione progrediuntur.

Solutio.

Quoniam sinus angulorum arithmeticam progreſſionem conſtituentium ſeriem recurrentem præbent, caſus hic ad lemma ſecundum pertinet, eritque $m = 2\cos Au$ & $n = -1$. Porro erit $A = \sin As$, $\Delta = \sin A(s-u)$; $P = \sin A(s+(p-1)u)$ & $= Q \sin A(s+pu)$. Ex his reperietur ſeriei propoſitæ ſumma

$$= \frac{\alpha(\sin As - \sin A(s-u))}{2-2\cos Au} - \frac{\alpha+p\beta)\ (\sin A(s+pu) - \sin A(s+(p-1)u))}{2 - \cos A u}$$

$$+ \frac{2\beta\ (\cos Au - 1)\ (\sin As - \sin A(s+pu))}{4\ (1-\cos Au)^2}$$

$$= \frac{\alpha}{2}\sin As + \frac{\alpha\cos As}{2\operatorname{tang}\frac{1}{2}u} - \frac{(\alpha+p\beta)}{2}\sin A(s+pu) -$$

$$\frac{(\alpha+p\beta)\ \cos A(s+pu)}{2\operatorname{tang}\frac{1}{2}u} - \frac{\beta\sin As + \beta\sin A(s+pu).}{2(1-\cos Au)}$$

Quæ ſumma reducitur ad hanc formam $=$

$$\frac{\alpha\cos A(s-\frac{1}{2}u) - (\alpha+p\beta)\cos A(s+(p-\frac{1}{2})u)}{2\sin A\frac{1}{2}u}$$

$$\frac{-\beta\sin As + \beta\sin A(s+pu).}{2(1-\cos Au)} \quad \text{Q. E. J.}$$

Co-

Coroll. 1.

23. Quoniam eſt $1 - \cos A u = 2(\sin A \tfrac{1}{2} u)^2$ erit quoque ſeriei ſinuum propoſitorum ſumma $=$

$$\frac{\alpha \cos A(s-\tfrac{1}{2}u)-(\alpha+p\beta)\cos A(s+(p-\tfrac{1}{2})u)}{2\sin A \tfrac{1}{2} u}-\frac{\beta \sin A s+\beta \sin(s+pu)}{4(\sin A \tfrac{1}{2} u)^2}$$

Coroll. 2.

24. Si ergo ſit $\alpha = 1$ & $\beta = 1$ erit hujus ſeriei:

$\sin A s + 2\sin A(s+u) + 3\sin A(s+2u) + \ldots\ldots + p\sin A(s+(p-1)u)$

$$\text{ſumma} = \frac{\cos A(s-\tfrac{1}{2}u)-(p+1)\cos A(s+(p-\tfrac{1}{2})u)}{2\sin A\tfrac{1}{2}u}-\frac{\sin A s+\sin A(s+pu)}{4(\sin A\tfrac{1}{2}u)^2}$$

Scholion.

25. Hæc eadem ſumma ſine ſubſidio lemmatis ex ſumma coſinuum ſimplicium ante inventa, ope differentiationis erui poteſt: Cum enim ſit

$\cos A s + \cos A(s+u) + \cos A(s+2u) + \ldots\ldots + \cos A(s+(p-1)u)$

$= \dfrac{-\sin A(s-\tfrac{1}{2}u)+\sin A(s+(p-\tfrac{1}{2})u)}{2\sin A\tfrac{1}{2}u}$; ponatur $s = a + \alpha x$ & $u = \beta x$ erit $\cos A(a+\alpha x) + \cos A(a+(\alpha+\beta)x) + \cos A(a+(\alpha+2\beta)x) + \ldots\ldots + \cos A(a+(\alpha+(p-1)\beta)x) =$

$\dfrac{-\sin A(a+(\alpha-\tfrac{1}{2}\beta)x)+\sin A(a+(\alpha+p\beta-\tfrac{1}{2}\beta)x)}{2\sin A\tfrac{1}{2}\beta x}$. Jam differentietur hæc æquatio poſito x variabili, & diviſione per $-dx$ facta habebitur:

$\alpha\sin A s + (\alpha+\beta)\sin A(s+u) + \ldots\ldots + (\alpha+(p-1)\beta)\sin A(s+(p-1)u)$

$$= \frac{(\alpha-\tfrac{1}{2}\beta)\cos A(s-\tfrac{1}{2}u)-(\alpha+(p-\tfrac{1}{2})\beta)\cos A(s+(p-\tfrac{1}{2})u}{2\sin A\tfrac{1}{2}u}$$

$$\frac{-\frac{1}{2}\beta \operatorname{cos} A\,\frac{1}{2}u.\operatorname{sin} A\,(s-\frac{1}{2}u) + \frac{1}{2}\beta \operatorname{cos} A\,\frac{1}{2}u.\operatorname{sin} A\,(s+(p-\frac{1}{2})u)}{2\,(\operatorname{sin} A\,\frac{1}{2}u)^2}$$

quæ expreſſio facile ad formam primum inventam reducitur.

Coroll. 3.

26. Ex inventa ſumma ſeriei ſinuum propoſitæ per differentiationem poſito u conſtante & s variabili orietur ſumma ſimilis ſeriei coſinuum:

$$\alpha \operatorname{cos} A\,s + (\alpha+\beta)\operatorname{cos} A\,(s+u) + (\alpha+2\beta)\operatorname{cos} A\,(s+2u) + \ldots$$

$$\ldots\ldots + (\alpha+(p-1)\beta)\operatorname{cos} A\,(s+(p-1)u) = \frac{-\alpha \operatorname{sin} A\,(s-\frac{1}{2}u)}{2\operatorname{sin} A\,\frac{1}{2}u}$$

$$+\frac{(\alpha+p\beta)\operatorname{sin} A\,(s+(p-\frac{1}{2}u)}{2\operatorname{sin} A\,\frac{1}{2}u} - \frac{\beta \operatorname{cos} A + \beta \operatorname{cos} A\,s\,(s+pu)}{4\,(\operatorname{sin} A\,\frac{1}{2}u)^2}$$

Lemma 3.

27. Hujus formulæ differentialis $\frac{x^{m-1}\,dx}{1+x^{2n}}$, in qua m eſt numerus minor quam $2n$, integrale eſt

$$\mp \frac{1}{2n} \operatorname{cos} A\,\frac{m\pi}{2n}\, l\,(1+2x\operatorname{cos} A\,\frac{\pi}{2n} + xx) \mp \frac{1}{n}\operatorname{sin} A\,\frac{m\pi}{2n}\,\text{Atang.}\,\frac{x\operatorname{sin} A\,\frac{\pi}{2n}}{1+x\operatorname{cos} A\,\frac{\pi}{2n}}$$

$$\mp \frac{1}{2n} \operatorname{cos} A\,\frac{3m\pi}{2n}\, l\,(1+2x\operatorname{cos} A\,\frac{3\pi}{2n} + xx) \mp \frac{1}{n}\operatorname{sin} A\,\frac{3m\pi}{2n}\,\text{A tang.}\,\frac{x\operatorname{sin} A\,\frac{3\pi}{2n}}{1+x\operatorname{cos} A\,\frac{3\pi}{2n}}$$

$\mp$

$$\mp \frac{1}{2n} \operatorname{cos} A \frac{5m\pi}{2n} l \left(1 + 2x \operatorname{cos} A \frac{5\pi}{2n} + xx\right) \mp \frac{1}{n} \operatorname{sin} A \frac{5m\pi}{2n} A \operatorname{tang.} \frac{x \operatorname{sin} A \frac{5\pi}{2n}}{1 + x \operatorname{cos} A \frac{5\pi}{2n}}$$

$$\vdots \qquad \vdots$$

$$\mp \frac{1}{2n} \operatorname{cos} A \frac{(2n-1)m\pi}{2n} l \left(1 \overline{+2} x \operatorname{cos} A \frac{(2n-1)\pi}{2n} + xx\right) \mp \frac{1}{n} \operatorname{sin} A \frac{(2n-1)m\pi}{2n} A \operatorname{tang.} \frac{x \operatorname{sin} A \frac{(2n-1)\pi}{2n}}{1 + x \operatorname{cos} A \frac{(2n-1)\pi}{2n}}$$

ubi ſigna ſuperiora valent ſi m fuerit numerus par, inferiora autem ſi m ſit numerus impar. Atque integrale hoc ita eſt acceptum, ut evanescat poſito $x = 0$.

Coroll. 1.

28. Hujus ergo formulæ differentialis $\frac{x^{2n-m-1} dx}{1 + x^{2n}}$ integraele erit

$$\mp \frac{1}{2n} \operatorname{cos} A \frac{m\pi}{2n} l \left(1 + 2x \operatorname{cos} A \frac{\pi}{2n} + xx\right) \mp \frac{1}{n} \operatorname{sin} A \frac{m\pi}{2n} A \operatorname{tang.} \frac{x \operatorname{sin} A \frac{\pi}{2n}}{1 + x \operatorname{cos} A \frac{\pi}{2n}}$$

$$\pm\frac{1}{2n}\cos A\frac{3m\pi}{2n}\, l\left(1+2x\cos A\frac{3\pi}{2n}+xx\right)\mp\frac{1}{n}\sin A\frac{3m\pi}{2n}\mathrm{Atang.}\frac{x\sin A\frac{3\pi}{2n}}{1+x\cos A\frac{3\pi}{2n}}$$

$$\vdots \qquad\qquad \vdots$$

$$\pm\frac{1}{2n}\cos A\frac{(2n-1)m\pi}{2n}\, l\left(1+2x\cos A\frac{(2n-1)\pi}{2n}+xx\right)\mp\frac{1}{n}\sin A\frac{(2n-1)m\pi}{2n}\,\mathrm{A\ tang.}\frac{x\sin A\frac{(2n-1)\pi}{2n}}{1+x\cos A\frac{(2n-1)\pi}{2n}}$$

ubi iterum ſigna ſuperiora valent, ſi m ſit numerus par, inferiora vero, ſin ſit numerus impar.

Coroll. 2.

29. Si ergo hæ ſormulæ differentiales addantur, in earum integrali quantitates logarithmicæ ſe deſtruunt, arcus circulares autem duplicabuntur, eritque ideo $\int\frac{x^{m-1}+x^{2n-m-1}}{1+x^{2n}}dx=$

$$\mp\frac{2}{n}\sin A\frac{m\pi}{2n}\mathrm{A\,tang.}\frac{x\sin A\frac{\pi}{2n}}{1+x\cos A\frac{\pi}{2n}}\mp\frac{2}{n}\sin A\frac{3m\pi}{2n}\mathrm{A\ tang.}\frac{x\sin A\frac{3\pi}{2n}}{1+x\cos A\frac{3\pi}{2n}}$$

$$\mp \frac{2}{n} \sin A \frac{5m\pi}{2} A \tan g. \frac{x \sin A \frac{5\pi}{2n}}{1 + x \cos A \frac{5\pi}{2n}} \mp - - - -$$

$$- - - \mp \frac{2}{n} \sin A \frac{(2n-1)m\pi}{2n} A \tan g. \frac{x \sin A \frac{(2n-1)\pi}{2n}}{1 + x \cos A \frac{(2n-1)\pi}{2n}}$$

ubi ſigna ſuperiora valent, ſi m fuerit numerus par, inferiora autem, ſin m ſit impar: denotatque perpetuo π arcum 180° in circulo cujus radius $= 1$.

Problema. 6.

30. Invenire integrale formulæ differentialis $\frac{x^{m-1} dx}{1 + x^{2n}}$, caſu quo poſt integrationem ponitur $x = \infty$.

Solutio.

Si in partibus integralis ante exhibiti logarithmicis ponatur $x = \infty$. exabibunt in $\mp \frac{lx}{n} (\cos A \frac{m\pi}{2n} + \cos A \frac{3m\pi}{2n} + \cos A \frac{5m\pi}{2n} + \ldots\ldots + \cos A \frac{(2n-1)m\pi}{2n})$ quorum arcuum cum differentia conſtans ſit $= \frac{2m\pi}{2n}$ erit horum coſinuum ſumma $=$

$$\frac{-\sin A\, 0\, \pi + \sin A \frac{2nm\pi}{2n}}{2 \sin A \frac{m\pi}{2n}} = 0;$$ etſi ergo x eſt infinitum tamen ejus logarithmus lx eſt ex minimo infinitorum ordine, hincque fit $0\, lx = 0$; Caſu ergo $x = \infty$ in integrali

 omnia

omnia membra a logarithmis pendentia ſe deſtruunt; ac remanebunt tantum altera membra a quadratura circuli pendentia. Cum vero ob x infinitum fiat

$$A\,\text{tang.}\frac{x \sin A \frac{k\pi}{2n}}{1 + x \cos A \frac{k\pi}{2n}} = A\,\text{tang.}\frac{\sin A \frac{k\pi}{2n}}{\cos A \frac{k\pi}{2n}} = \frac{k\pi}{2n}$$

erit integrale quæſitum caſu $x = \infty$; $=$

$$\mp \frac{\pi}{2nn}\left(\sin A \frac{m\pi}{2n} + 3 \sin A \frac{3m\pi}{2n} + 5 \sin A \frac{5m\pi}{2n} + \ldots\ldots + (2n-1) \sin A \frac{(2n-1)m\pi}{2n}\right)$$

Quæ ſeries ſinuum per probl. 5. in unam ſummam colligi poterit. Erit autem $\alpha = 1$, $\beta = 2$; $p = n$; deinde $s = \frac{m\pi}{2n}$; $u = \frac{2m\pi}{2n}$ & $\frac{1}{2}\ u = \frac{m\pi}{2n}$ ex quibus hujus ſeriei ſinuum ſumma colligitur eſſe

$$= \frac{1 - (2n+1)\cos A\, m\pi}{2 \sin A \frac{m\pi}{2n}} - \frac{2 \sin A \frac{m\pi}{2n} + 2 \sin A \frac{(2n+1)m\pi}{2n}}{4\left(\sin A \frac{m\pi}{2n}\right)^2}$$

$$= \frac{\sin A \left(m\pi + \frac{m\pi}{2n}\right)}{2\left(\sin A \frac{m\pi}{2n}\right)^2} - \frac{(2n+1)\cos A\, m\pi}{2 \sin A \frac{m\pi}{2n}} = \frac{-n \cos A\, m\pi}{\sin A \frac{m\pi}{2n}}$$

Quodſi jam fuerit m numerus par, erit $\cos A\, m\pi = +1$ ſin autem m ſit numerus impar, erit $\cos A\, m\pi = -1$ Signis ambiguis igitur ſumma ſuperior ſinuum ita exprimetur ut ſit $= \mp \frac{n}{\sin A \frac{m\pi}{2n}}$, quæ ducta in $\mp\ \frac{\pi}{2nn}$ dabit,

five

ſive m ſit numerus par ſive impar eandem integralis quæſiti quantitatem $= \frac{\pi}{2n \sin A \frac{m\pi}{2n}}$ ad hancque expreſſionem reducitur integrale hujus formulæ $\frac{x^{m-1}\, dx}{1 + x^{2n}}$, ſi poſt integrationem ponatur $x = \infty$. Q. E. J.

Coroll. 1.

32. Erit igitur $\int \frac{x^{p-1}\, dx}{1+x^q} = \frac{\pi}{q \sin A \frac{p\pi}{q}}$ poſt integrationem poſito $x = \infty$, ſiquidem q fuerit numerus par; & exponens p minor exponente q.

Scholion.

32. Ut autem appareat, quemnam valorem habitura ſit formula $\int \frac{x^{p-1}\, dx}{1+x^q}$, ſi q fuerit numerus impar, poſito poſt integrationem $x = \infty$; ponamus $x = y\, y$, atque formula noſtra transibit in hanc $2 \int \frac{y^{2p-1}\, dy}{1+y^{2q}}$, qui caſus cum contineatur in propoſito, erit ejus valor poſito $y = \infty$ quo facto ſimul x ſit infinitum, $= 2 . \frac{\pi}{2q \sin A \frac{p\pi}{q}}$ erit ergo quoque $\int \frac{x^{p-1}\, dx}{1+x^q} = \frac{\pi}{q \sin A \frac{p\pi}{q}}$, poſito poſt integrationem

tionem $x = \infty$, ſi q fuerit numerus impar, Generaliter ergo quicunque fuerint numeri p & q dummodo $p-1$ ſit minor quam q, erit ſemper $\int \frac{x^{p-1}\,dx}{1+x^q} = \frac{\pi}{q \sin A \frac{p\pi}{q}}$.

Oportet autem eſſe $p-1 < q$, quia alias integrale lemmate 3 datum non eſſet completum, verum inſuper membrum unum plurave algebraica reciperet, ob quæ integrale caſu $x = \infty$ ſemper fieret infinitum.

Coroll. 2.

33. Si ponamus $x = \frac{y}{(1-y^q)^{\frac{1}{q}}}$, erit $x = 0$ ſi $y = 0$ & $x = \infty$ ſi ponatur $y = 1$; tum autem fiet $dx = \frac{dy}{(1-y^q)^{\frac{1+q}{q}}}$; $1 + x^q = \frac{1}{1-y^q}$ & $x^{p-1} = \frac{y^{p-1}}{(1-y^q)^{\frac{p-1}{q}}}$, unde erit $\frac{x^{p-1}\,dx}{1+x^q} = \frac{y^{p-1}\,dy}{(1-y^q)^{\frac{p}{q}}}$.

Quocirca integrando fiet $\int \frac{y^{p-1}\,dy}{(1-y^q)^{\frac{p}{q}}} = \frac{\pi}{q \sin A \frac{p}{q}\pi}$ ſi poſt integrationem ponatur $y = 1$.

Problema. 7.

Problema. 7.

34. Invenire integrale formulæ differentialis

$\frac{x^{p-1}\,dx}{(1+x^q)^k}$, caſu quo poſt integrationem ponitur $x = \infty$.

Solutio.

Per reductionem formularum integralium erit $\int \frac{x^{p-1}\,dx}{(1+x^q)^k}$

$$= \frac{x^p}{(k-1)q(1+x^q)^{k-1}} + \frac{(k-1)q-p}{(k-1)q}\int\frac{x^{p-1}\,dx}{(1+x^q)^{k-1}}$$, ſi ergo

poſt integrationem, uti aſſumimus, ponatur $x = \infty$, membrum algebraicum ob $p < q\,(k-1)$ evanescit eritque:

$$\int \frac{x^{p-1}\,dx}{(1+x^q)^k} = \frac{(k-1)q-p}{(k-1)q} \int \frac{x^{p-1}\,dx}{(1+x^q)^{k-1}}.$$ Quamobrem

ſi loco k ſucceſſive ponamus numeros 2, 3, 4, 5, &c. omnes hæ formulæ integrales reducentur ad hanc

$\int \frac{x^{p-1}\,dx}{1+x^q}$, cujus valorem caſu $x = \infty$ vidimus eſſe

$= \frac{\pi}{q \text{ ſi A } \frac{p\pi}{q}}$; unde ſequentes nascentur integrationes.

$$\int \frac{x^{p-1}\,dx}{(1+x^q)^2} = \frac{q-p}{p} \cdot \frac{\pi}{q \text{ ſin A } \frac{p\pi}{q}}$$

$$\int \frac{x^{p-1}\,dx}{(1+x^q)^3} = \frac{(q-p)\;(2q-p)}{q\;.\;\;2q}\cdot\frac{\pi}{q\,\sin A\frac{p\pi}{q}}$$

$$\int \frac{x^{p-1}\,dx}{(1+x^q)^4} = \frac{(q-p)\;(2q-p)\;(3q-p)\,.}{q\;.\;\;2q\;.\;\;3q}\;\frac{\pi}{q\,\sin A\frac{p\pi}{q}}$$

&c.

Hincque generaliter concludetur fore

$$\int \frac{x^{p-1}\,dx}{(1+x^q)^k} = \frac{(q-p)(2q-p)(3q-p)\ldots\,((k-1)q-p)}{q\;.\;\;2q\;.\;\;3q\;\ldots\;\;(k-1)q}\cdot\frac{\pi}{q\,\sin A\frac{p\pi}{q}}$$

Q. E. J.

Coroll. 1.

35. Quoties ergo k fuerit numerus integer affirmativus, toties integrale formulæ $\int \frac{x^{p-1}\,dx}{(1+x^q)^k}$ casu quo $x = \infty$, per peripheriam circuli exprimi potest.

Coroll. 2.

36. Ex dissertatione autem mea de progressionibus transcendentibus Tom. Comment. IV. colligitur esse

$$\frac{q\;.\;\;2q\;.\;\;3q\;\;\ldots\ldots\;\;(k-1)q}{(q-p)(2q-p)(3q-p)(\;\ldots\ldots\;((k-1)q-p)} = (kq-p)\int y^{q-p-1}\,dy$$

$(1-y^q)^{k-1}$ posito post integrationem $y = 1$. Hinc ergo colligitur fore $\int \frac{x^{p-1}\,dx}{(1+x^q)}\,.\;\int y^{q-p-1}\,dy\;(1-y^q)^{k-1}$

=

$$= \frac{\pi}{q(kq-p)\,\text{fin}\,A\frac{p\pi}{q}}.$$

Coroll. 3.

37. Si ponamus $x = \frac{y}{(1-y^q)^{\frac{1}{q}}}$, ita ut posito $y = 1$ fiat $x = \infty$, fiet $\int \frac{x^{p-1}\,dx}{(1+x^q)^k} = \int y^{p-1}\,dy(1-y^q)^{\frac{(k-1)q-p}{q}}$ posito ergo $y = 1$, fiet $\frac{\pi}{q(kq-p)\,\text{fin}\,A\frac{p\pi}{q}} =$

$\int y^{q-p-1}\,dy\,(1-y^q)^{k-1}.\int y^{p-1}\,dy\,(1-y^q)^{\frac{(k-1)q-p}{q}}$. At est $\int y^{p-1}\,dy(1-y^q)^{\frac{(k-1)q-p}{q}} = \frac{kq}{kq-p}\int y^{p-1}\,dy\,(1-y^q)^{\frac{kq-p}{q}}$ ex quo erit $\int y^{q-p-1}\,dy(1-y^q)^{k-1}.\int y^{p-1}\,dy(1-y^q)^{\frac{kq-p}{q}}$

$$= \frac{\pi}{kqq\,\text{fin}\,A\frac{p\pi}{q}}.$$

Problema 8.

38. Invenire integrale formulæ differentialis $\frac{x^{m-1}+x^{2n-m-1}}{1+x^{2n}}\,dx$, casu quo post integrationem ponitur $x = 1$.

Solutio.

Solutio.

Hujus formulæ differentialis integrale in genere exhibuimus (29). Posito autem $x = 1$, quælibet forma a quadratura circuli pendens A tang. $\frac{x \sin A\, \Phi}{1 + x \cos A\, \Phi}$ abit in $\frac{\Phi}{2}$. Hinc formulæ propositæ integrale casu $x = 1$ erit $= \mp$
$\frac{\pi}{2nn}$ (sin A$\frac{m\pi}{2n}$ + 3 sin A $\frac{3m\pi}{2n}$ + 5 sin A $\frac{5m\pi}{2n}$ + &c.
...... + $(2n-1)$ sin A $\frac{(2n-1)m\pi}{2n}$) quæ est illa ipsa sinuum series, quam in solutione probl. 6. ad summam definitam revocavimus; ubi pariter signa superiora valent si m fuerit numerus par, inferiora sin m numerus impar. Utroque ergo casu sive m sit numerus par sive impar integrale quæsitum erit idem, quod in problemate sexto: scilicet posito post integrationem $x = 1$ erit $\int \frac{x^{m-1} + x^{2n-m-1}}{1 + x^{2n}} dx =$

$\frac{\pi}{2n \sin A \frac{m\pi}{2n}}$ Q. E. J.

Coroll. 1.

39. Erit ergo $\int \frac{x^{p-1} + x^{q-p-1}}{1 + x^q} dx = \frac{\pi}{q \sin A \frac{p\pi}{q}}$
posito post integrationem $x = 1$; siquidem fuerit exponens $p-1$ minor exponente; uti supra annotavimus.

Scholion.

40. Simili modo, quo supra (32) usi sumus, ostendi potest

eundem

eundem integralis valorem locum obtinere, etiamsi q sit numerus impar, sit enim in formula $\int \frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx$ exponens q numerus impar; ponamusque $x = yy$, abibit hæc formula in hanc $2\int \frac{y^{2p-1}+y^{2q-2p-1}}{1+y^{2q}}dy$, cujus utique integrale casu quo $y = 1$ erit $=$ $2. \frac{\pi}{2q \sin A \frac{p\pi}{q}}$; erit ergo sive q sit numerus par sive impar $\int \frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx = \frac{\pi}{q \sin A \frac{p\pi}{q}}$.

Coroll. 2.

41. Integralia igitur harum duarum formularum differentialium $\int \frac{x^{p-1}dx}{1+x^q}$ & $\int \frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx$ si in illa post integrationem ponatur $x = \infty$ in hac autem $x = 1$, erunt inter se æqualia, utroque scilicet casu integrale est $= \frac{\pi}{q \sin A \frac{p}{q}\pi}$.

Coroll. 3.

42. Si in integrali $\int \frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx$ ponatur integra-

tione

tione peracta $x = \infty$, erit ejus valor $=$

$$\frac{\pi}{q \sin A \frac{p\pi}{q}} + \frac{\pi}{q \sin A \frac{(q-p)\pi}{q}} = \frac{2\pi}{q \sin A \frac{p\pi}{q}};$$ hoc ergo integrale duplo majus est; si ponatur $x = \infty$ quam si ponatur $x = 1$.

Lemma 4.

43. Hujus formulæ differentialis $\frac{x^{m-1}\, dx}{1 - x^{2n}}$ in qua $m - 1$ est numerus minor quam $2n$, integrale est:

$$\mp \frac{l(1+x) - l(1-x)}{2n}.$$

$$\mp \frac{1}{2n} \cos A \frac{m\pi}{n}\, l\left(1 + 2x \cos A \frac{\pi}{n} + xx\right) \pm \frac{1}{n} \sin A \frac{m\pi}{n}\, A\,\text{tang.}\, \frac{x \sin A \frac{\pi}{n}}{1 + x \cos A \frac{\pi}{n}}$$

$$\mp \frac{1}{2n} \cos A \frac{2m\pi}{n}\, l\left(1 + 2x \cos A \frac{2\pi}{n} + xx\right) \pm \frac{1}{n} \sin A \frac{2m\pi}{n}\, A\,\text{tang.}\, \frac{x \sin A \frac{2\pi}{n}}{1 + x \cos A \frac{2\pi}{n}}$$

$$\pm \frac{1}{2n} \cos A \frac{3m\pi}{2n}\, l\left(1 + 2x \cos A \frac{3\pi}{2n} + xx\right) \pm \frac{1}{n} \sin A \frac{3m\pi}{n}\, A\,\text{tang.}\, \frac{x \sin A \frac{3\pi}{n}}{1 + x \cos A \frac{3\pi}{n}}$$

. . . .

$$\vdots \qquad\qquad\qquad \vdots$$

$$\pm \frac{1}{2n} \cos A \frac{(n-1)m\pi}{n} \, l\left(1+2x\cos A\frac{(n-1)\pi}{n} + xx\right) \pm \frac{1}{n} \sin A \frac{(n-1)m\pi}{n} A \text{ tang.} \frac{x \sin A \frac{(n-1)\pi}{n}}{1+x\cos A \frac{(n-1)\pi}{n}}$$

ubi signa superiora valent, si m est numerus impar, inferiora autem, si m fuerit numerus impar.

Coroll. 1.

44. Hinc istius formulæ differentialis $\frac{x^{2n-m-1}\,dx}{1-x^{2n}}$ integrale erit sequens:

$$\pm \frac{1}{2n} \, l\,(1+x) - \frac{1}{2n} \, l\,(1-x)$$

$$\pm \frac{1}{2n} \cos A \frac{m\pi}{n} \, l\left(1+2x\cos A\frac{\pi}{n} + xx\right) \mp \frac{1}{n} \sin A \frac{m\pi}{n} A \text{ tang.} \frac{x \sin A \frac{\pi}{n}}{1+x\cos A\frac{\pi}{n}}$$

$$\pm \frac{1}{2n} \cos A \frac{2m\pi}{n} \, l\left(1+2x\cos A\frac{2\pi}{n} + xx\right) \mp \frac{1}{n} \sin A \frac{2m\pi}{n} A \text{ tang.} \frac{x \sin A \frac{2\pi}{n}}{1+x\cos A\frac{2\pi}{n}}$$

$\pm$

$$\pm\frac{1}{2n}\cos A\frac{3m\pi}{n}\,l\left(1+2x\cos A\frac{3\pi}{2n}+xx\right)\mp\frac{1}{n}\sin A\frac{3m\pi}{n}\,A\,\text{tang.}\frac{x\sin A\frac{3\pi}{n}}{1+x\cos A\frac{3\pi}{n}}$$

$$\vdots \qquad\qquad \vdots$$

$$\pm\frac{1}{2n}\cos A\frac{(n-1)m\pi}{n}\,l\left(1+2x\cos A\frac{(n-1)\pi}{n}+xx\right)\mp\frac{1}{n}\sin A\frac{(n-1)m\pi}{n}\,A\,\text{tang.}\frac{x\sin A\frac{(n-1)\pi}{n}}{1+x\cos A\frac{(n-1)\pi}{n}}$$

ubi iterum ſigna ſuperiora locum habent, ſi m ſit numerus impar, inferiora vero, ſi m ſit numerus par.

Coroll. 2.

45. Si igitur hæc formula poſterior a priori ſubtrahatur, membra logarithmica ſe mutuo deſtruent, eritque hujus formulæ $\frac{x^{m-1}+x^{2n-m-1}}{1+x^{2n}}dx$ integrale $=$

$$\pm\frac{2}{n}\sin A\frac{m\pi}{n}\,A\,\text{tang.}\frac{x\sin A\frac{\pi}{n}}{1+x\cos A\frac{\pi}{n}}\mp\frac{2}{n}\sin A\frac{2m\pi}{n}\,A\,\text{tang.}\frac{x\sin A\frac{2\pi}{n}}{1+x\cos A\frac{2\pi}{n}}$$

$$\mp \frac{2}{n} \sin A \frac{3m\pi}{n} A\,tang. \frac{x \sin A \frac{3\pi}{n}}{1 + x \cos A \frac{3\pi}{n}} \pm - \quad -$$

$$- \quad - \quad - \qquad \frac{2}{n} \sin A \frac{(n-1)m\pi}{n} A\,tang. \frac{x \sin A \frac{(n-1)\pi}{n}}{1 + x \cos A \frac{(n-1)\pi}{n}}$$

ubi ſignorum ambiguorum valor ſe habet ut ante.

Problema. 9.

46. Invenire integrale hujus formulæ differentialis $\frac{x^{m-1} - x^{2n-m-1}}{1 - x^{2n}} dx$, eo caſu, quo poſt integrationem abſolutam ponitur $x = 1$.

Solutio.

Quoniam caſu $x = 1$ eſt $A\,tang. \frac{x \sin A \Phi}{1 + x \cos A \Phi} = A\,tang. \frac{\sin A \frac{1}{2}\Phi}{\cos A \frac{1}{2}\Phi} = \frac{1}{2}\Phi$, erit quæſitum integrale $= \pm \frac{\pi}{nn} \Big(\sin A \frac{m\pi}{n} + 2 \sin A \frac{2m\pi}{n} + 3 \sin A \frac{3m\pi}{n} + \ldots\ldots\ldots + (n-1) \sin A \frac{(n-1)m\pi}{n} \Big)$ ubi ſignorum ambiguorum ſuperius $+$ valet, ſi m ſit numerus impar, inferius vero, ſi m ſit par. Hujus ergo ſeriei ſinuum ſumma per probl. 5. reperietur, eritque facta applicatione $\alpha = 1$, $\beta = 1$, $p = n - 1$; $s = \frac{m\pi}{n}$; $u = \frac{m\pi}{n}$ & $\frac{1}{2} u = \frac{m\pi}{2n}$ hinc ſumma quæſita erit $=$

Y cos.

$$\frac{\cos A\frac{m\pi}{2n} - n\cos A\left(m\pi - \frac{m\pi}{2n}\right)}{2\sin A\frac{m\pi}{2n}}.$$

$$\frac{-\sin A\frac{m\pi}{2n} + \sin A\, m\,\pi}{4\left(\sin A\frac{m\pi}{2n}\right)^2}.$$ Quia vero eſt $\sin A\frac{m\pi}{n} = 2\sin A\frac{m\pi}{2n}.\cos A\frac{m\pi}{2n}$; $\sin A\, m\pi = 0$ & $\cos A\left(m\,\pi - \frac{m\pi}{2n}\right) = \cos A\, m\,\pi.\cos A\frac{m\pi}{2n}$; erit ſumma ſeriei ſinuum inventæ $= \frac{-n\cos A\, m\pi.\ \cos A\frac{m\pi}{2n}}{2\sin A\frac{m\pi}{2n}} = \frac{\pm n\cos A\frac{m\pi}{2n}}{2\sin A\frac{m\pi}{2n}}$, ubi, ut ante, ſignum ſuperius locum habet, ſi m ſit numerus impar, inferius vero, ſi m ſit numerus par. Hoc modo ſignorum ambiguitas tollitur, eritque formulæ differentialis propoſitæ integrale caſu $x=1$, ſive m ſit numerus impar ſive par, conſtanter $= \frac{\pi\cos A\frac{m\pi}{2n}}{2n\sin A\frac{m\pi}{2n}}$. Q. E. J.

Coroll. 1.

47. Si igitur poſt integrationem ita abſolutam ut integrale evaneſcat poſito $x = 0$ ponatur $x = 1$ erit

∫

$$\int \frac{x^{p-1} - x^{q-p-1}}{1 - x^q} dx = \frac{\pi \operatorname{cof} A \frac{p\pi}{q}}{q \operatorname{fin} A \frac{p\pi}{q}},$$ fiquidem q fuerit numerus par.

Coroll. 2.

48. Eadem igitur integratio cafu faltem $x = 1$ locum quoque habebit, fi q fuerit numerus impar, cum pofito $x = yy$ exponens ipfius y in denominatore par reddatur. Erit ergo generaliter, fi poft integrationem $x = 1$ ponatur

$$\int \frac{x^{p-1} - x^{q-p-1}}{1 - x^q} dx = \frac{\pi \operatorname{cof} A \frac{p\pi}{q}}{q \operatorname{fin} A \frac{p\pi}{q}}.$$

Coroll. 3

49. Cum pofito pariter poft integrationem $x = 1$ fit

$$\int \frac{x^{p-1} + x^{q-p-1}}{1 + x^q} dx = \frac{\pi}{q \operatorname{fin} A \frac{p\pi}{q}},$$ fi illa per

hanc dividatur erit $\int \frac{x^{p-1} - x^{q-p-1}}{1 - x^q} dx = \operatorname{cof} A \frac{p\pi}{q}$.

$\int \frac{x^{p-1} + x^{q-p-1}}{1 + x^q} dx$, fiquidem poft utramque integrationem ponatur $x = 1$.

Lemma 5.

54. Hujus formulæ differentialis $\frac{x^{m-1} dx}{1 - 2bx^n + x^{2n}}$ fi ponatur

n nume-

n numerus par, ac ſtatuatur $n - m = i$ itemque capiatur angulus ω cujus coſinus ſit $= b$, erit integrale ſequens expreſſio:

$$\pm \frac{\text{ſin A}\,\frac{i}{n}\,\omega}{2n\,\text{ſin A}\,\omega}\, l\left(1 + 2\,x\,\text{coſA}\,\frac{\omega}{n} + x\,x\right) \pm \frac{\text{coſA}\,\frac{i}{n}\,\omega}{n\,\text{ſin A}\,\omega}\ \text{A tang.}\frac{x\ \text{ſin}\ A\,\frac{\omega}{n}}{1 + x\ \text{coſ A}\,\frac{\omega}{n}}$$

$$\pm \frac{\text{ſin A}\,\frac{i}{n}\,(2\pi + \omega)}{2\,n\,\text{ſin A}\,\omega}\, l\left(1 + 2x\,\text{coſ}\,A\,\frac{2\pi+\omega}{n} + x\,x\right) \pm \frac{\text{coſA}\,\frac{i}{n}(2\pi+\omega)}{n\,\text{ſin A}\,\omega}\ \text{A tang.}\frac{x\ \text{ſin A}\ \frac{2\pi+\omega}{n}}{1 + x\ \text{coſA}\frac{2\pi+\omega}{n}}$$

$$\pm \frac{\text{ſin}\,A\,\frac{i}{n}\,(4\pi + \omega)}{2n\ \text{ſin A}\ \omega}\, l\left(1 + 2\,x\,\text{coſA}\frac{4\pi+\omega}{n} + x\,x\right) \pm \frac{\text{coſA}\,\frac{i}{n}(4\pi+\omega)}{n\ \text{ſin A}\ \omega}\ \text{A tang.}\frac{x\ \text{ſin A}\ \frac{4\pi+\omega}{n}}{1 + x\,\text{coſA}\frac{4\pi+\omega}{n}}$$

$$\begin{array}{cc} \vdots & \vdots \\ \vdots & \vdots \end{array}$$

$\pm$

$$\pm \frac{\sin A\frac{i}{n}(2(n-1)\pi+\omega)}{2n\sin A\,\omega}\, l\left(1+2x\cos A\frac{2(n-1)\pi+\omega}{n}+xx\right) \pm \frac{\cos A\frac{i}{n}(2(n-1)\pi+\omega)}{n\sin A\,\omega}$$

$$A\,\text{tang.}\frac{x\sin A\,\frac{2(n-1)\pi+\omega}{n}}{1+x\cos A\frac{2(n-1)\pi+\omega}{n}}$$

ubi ſigna ſuperiora valent, ſi i eſt numerus impar, inferiora autem, ſi i ſit numerus par. Quod ſi autem n fuiſſet numerus impar, tum non ſolum hæc ſignorum lex debet commutari, ſed etiam pro ω debet capi angulus, cujus coſinus ſit $= -b$.

Problema 10.

51. Invenire integrale hujus formulæ differentialis $\frac{x^{m-1}\,dx}{1-2bx^n+x^{2n}}$ eo caſu, quo poſt integrationem ponitur $x = \infty$.

Solutio.

Integrale univerſaliter ſumtum conſtat duplici partium ordine, alter membra logarithmica complectitur, alter a quadratura circuli pendentia. Aſſumamus n eſſe numerum parem, ac ponamus $n-m = i$, ſitque ω arcus cujus cosinus $= b$. Posito jam $x = \infty$ ſinguli logarithmi abibunt in $lx^2 = 2lx$, horumque ideo membrorum logarithmicorum ſumma erit $\frac{\mp lx}{n\sin A\omega}\Big(\sin A\frac{i}{n}\,\omega + \sin A\frac{i}{n}\,(2\pi+\omega)$

$+ \ldots\ldots\ldots + \sin A\frac{i}{n}\,(2(n-1)\pi+\omega)\Big)$

quo-

quorum ſinuum omnium ſumma reperitur $=$

$$\frac{\operatorname{cof} A\,\frac{i}{n}\,(\omega-\pi) - \operatorname{cof} A\,\frac{i}{n}\,((2n-1)\,\pi+\omega)}{2\,\operatorname{fin} A\,\frac{i\pi}{n}}$$, cum autem

hi anguli differant integra peripheria 2π aliquoties ſumta, erunt eorum cosinus æquales, hincque ſumma omnium membrorum logarithmicorum in integrali $= 0$. Supererunt ergo tantum membra a quadratura circuli pendentia, quæ caſu $x = \infty$, ita ſehabebunt, $\pm \frac{1}{nn \operatorname{fin} A\omega}\,(\omega \operatorname{cof} A\,\frac{i}{n}\,\omega +$

$(2\pi+\omega)\,\operatorname{cof} A\,\frac{i}{n}\,(2\pi+\omega) + (4\pi+\omega)\operatorname{cof} A\,\frac{i}{n}(4\pi+\omega)$

$+ \ldots + (2(n-1)\,\pi+\omega)\,\operatorname{cof} A\,\frac{i}{n}\,(2\,(n-1)\,\pi+\omega))$

Hæc jam coſinuum ſeries per (26) ſummabitur, factaque comparatione erit $\alpha = \omega$; $\beta = 2\pi$; $s = \frac{i\omega}{n}$ $u = \frac{2i\pi}{n}$, & $p = n$, unde obtinetur ſumma horum cosinuum $=$

$$\frac{-\omega \operatorname{fin} A\frac{i}{n}\,(\omega-\pi) + (\omega+2n\pi)\operatorname{fin} A\,\frac{i}{n}\,(\omega+(2n-1)\pi)}{2\operatorname{fin} A\,\frac{i\pi}{n}}$$

$$\frac{-\pi \operatorname{cof} A\frac{i\omega}{n} + \pi \operatorname{cof} A\frac{i}{n}\,(\omega+2n\pi)}{2\left(\operatorname{fin} A\frac{i\pi}{n}\right)^2} = \frac{n\pi \operatorname{fin} A\frac{i}{n}\,(\omega-\pi)}{\operatorname{fin} A\frac{i\pi}{n}}$$

Integrale

Integrale ergo quæsitum eſt $= \pm \dfrac{\pi \sin A \frac{i}{n} (\omega - \pi)}{n \sin A\omega . \sin A \frac{i\pi}{n}}$

ſublata autem ſignorum ambiguitate erit formulæ differentialis propoſitæ integrale caſu $x = \infty$, hoc

$$\int \frac{x^{m-1} dx}{1 - 2bx^n + x^{2n}} = \frac{\pi \sin A \frac{i}{n} (\pi - \omega)}{n \sin A\omega . \sin A \frac{i\pi}{n}},$$ exiſtente $i = n - m$ & $\omega = A \cos b$. Q. E. J.

Coroll. 1.

52. Si loco m ſcribatur $2n - m$ tum i in ſui negativum abit, quo ipſo integrale non afficitur, erit ergo

$$\int \frac{x^{m-1} dx}{1 - 2bx^n + x^{2n}} = \int \frac{x^{2n-m-1} dx}{1 - 2bx^n + x^{2n}}$$ caſu quo ponitur $x = \infty$.

Coroll. 2.

53. Si fiat $m = n$ tum erit $i = 0$; quo caſu cum ſinus arcuum evaneſcentium ſint ipſis arcubus æquales, fiet

$$\int \frac{x^{n-1} dx}{1 - 2bx^n + x^{2n}} = \frac{\pi - \omega}{n \sin A\omega}$$ poſito $x = \infty$; cujus veritas facile poteſt comprobari.

Scholion.

54. Aſſumsimus hic b eſſe numerum unitate ſeu ſinu toto minorem, alioquin non daretur arcus ω, cujus coſinus eſſet $= b$.

$= b$. Ideo autem hunc casum præ reliquis elegi, quod denominator $1 - 2bx^n + x^{2n}$ non in duos factores reales binomiales resolvi potest. Quoties enim ejusmodi resolutio locum habet, facilius per præcedentia opus expediri potest.

Problema 11.

55. Invenire integrale hujus formulæ differentialis

$\frac{x^{p-1}\,dx}{1 + ax^q}$, casu tantum quo post integrationem ponitur $x = \infty$.

Solutio.

Ponatur $ax^q = y^q$ seu $x = a^{\frac{-1}{q}}\,y$, quo facto formula proposita abibit in hanc $\frac{a^{\frac{-p}{q}}\,y^{p-1}\,dy}{1 + y^q}$, cujus integrale casu $y = \infty$ est $= \frac{\pi}{a^{\frac{p}{q}}\,q\,\sin A\,\frac{p\pi}{q}}$. Cum autem posito $y = \infty$ simul fiat $x = \infty$ erit quoque hoc casu

$$\int \frac{x^{p-1}\,dx}{1 + ax^q} = \frac{\pi}{a^{\frac{p}{q}}\,q\,\sin A\,\frac{p\pi}{q}}.\quad \text{Q. E. J.}$$

Coroll. 1.

56. Erit igitur sumto quocunque multiplo

$$\int \frac{mx^{p-1}\,dx}{1 + ax^q} = \frac{m\pi}{a^{\frac{p}{q}}\,q\,\sin A\,\frac{p\pi}{q}}$$ si post integrationem ponatur $x = \infty$.

Coroll.

Coroll. 2.

37. Cum igitur simili modo sit $\int \frac{n x^{p-1} dx}{1 + b x^q} = \frac{n \pi}{b^{\frac{p}{q}} q \sin A \frac{p\pi}{q}}$, erit duas hujusmodi formulas addendo $\int \frac{(m+n) x^{p-1} dx + (mb + na) x^{p+q-1} dx}{1 + (a+b) x^q + ab x^{2q}} = \frac{\pi}{q \sin A \frac{p\pi}{q}} \left(\frac{m}{a^{\frac{p}{q}}} + \frac{n}{b^{\frac{p}{q}}} \right)$, posito post integrationem $x = \infty$.

Problema. 12.

58. Si ponatur post integrationem $x = \infty$, invenire valorem hujus integralis $\int \frac{x^{p-1} dx}{1 + 2 f x^q + g x^{2q}}$.

Solutio.

Comparata hac formula cum coroll. præced. fiet $2f = a + b$ & $g = ab$ unde $2\sqrt{(ff - g)} = a - b$, hincque $a = f + \sqrt{(ff - g)}$ & $b = f - \sqrt{(ff - g)}$. Porro autem erit $m + n = 1$ & $mb + na = 0$ seu $(m+n) f = (m-n) \sqrt{(ff-g)}$, ideoque $m - n = \frac{f}{\sqrt{(ff-g)}}$. Erit ergo $m = \frac{f + \sqrt{(ff-g)}}{2\sqrt{(ff-g)}}$ & $n = \frac{-f + \sqrt{(ff-g)}}{2\sqrt{(ff-g)}}$. His valoribus inventis obtinebitur integrale quæsitum $\int \frac{x^{p-1} dx}{1 + 2 f x^q + g x^{2q}}$

$$= \frac{\pi}{2q \sin A \frac{p\pi}{q}} \left(\frac{(f + \sqrt{(ff-g)})^{\frac{q-p}{q}} - (f - \sqrt{(ff-g)})^{\frac{q-p}{q}}}{\sqrt{(ff-g)}} \right)$$

si quidem post integrationem ponatur $x = \infty$. Q. E. J.

Coroll. 1.

59. Quodsi ergo f & g fuerint quantitates reales affirmativæ, atque fuerit $ff > g$, tum integrale inventum terminis realibus erit expressum. Sin autem g fuerit quantitas negativa, tum b erit negativum: hoc que casu integrale inventum locum habere nequit. Idem incommodum evenit; si f fuerit numerus negativus, existente $ff > g$, tum enim a & b fient numeri negativi, neque idcirco formularum simplicium $\frac{x^{p-1} dx}{1-ax^q}$ & $\frac{x^{p-1} dx}{1-bx^q}$ integralia casu $x = \infty$ exhiberi poterunt.

Coroll. 2.

60. Sin autem sit $g > ff$, tum utraque quantitas a & b fiet imaginaria, nisi igitur imaginaria in integrali invento se destruant, valor formulæ propositæ casu $x = \infty$ exhiberi non poterit.

Scholion.

61. Ponamus ergo esse $g > ff$, sitque ω angulus, cujus cosinus sit $= \frac{f}{\sqrt{g}}$, erit $\frac{\sqrt{(ff-g)}}{\sqrt{g}} = \sin A\omega . \sqrt{-1}$, hincque

$$(f + \sqrt{(ff-g)})^{\frac{q-p}{q}} = (\cos A\omega + \sqrt{-1} . \sin A\omega)^{\frac{q-p}{q}}$$

$g^{\frac{q-p}{2q}}$ & $(f - \sqrt{(ff-g)})^{\frac{q-p}{q}} = (\text{cos} A\,\omega - \sqrt{-1}.\,\text{sin}\, A\,\omega)^{\frac{q-p}{q}}$.
$g^{\frac{q-p}{2q}}$. At est $(\text{cos} A\,\omega \pm \sqrt{-1}.\,\text{sin} A\,\omega)^{\frac{q-p}{q}} = \text{cos} A \frac{(q-p)\,\omega}{q} \pm \sqrt{-1}.\,\text{sin}\, A \frac{(q-p)\,\omega}{q}$. Ex quibus formulæ propoſitæ $\frac{x^{p-1}\,dx}{1 + 2fx^q + xg^{2q}}$ caſu quo $g > ff$ integrale erit, ſi poſt integrationem $x = \infty$ ponatur $= \frac{\pi}{g^{\frac{p}{2q}q}\ \text{sin} A \frac{p\pi}{q}}$. $\frac{\text{sin} A \frac{(q-p)\,\omega}{q}}{\text{sin} A\,\omega}$ exiſtente $\text{cos} A\,\omega = \frac{f}{\sqrt{g}}$. Quod, ſi loco ω ſcribamus $\pi - \omega$, & i loco $q-p$ atque n loco q, congruit cum integrali pro eodem caſu in problemate 10 invento.

Fig: 1.

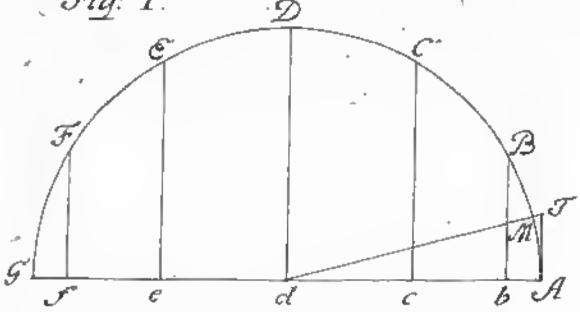


Fig: 3

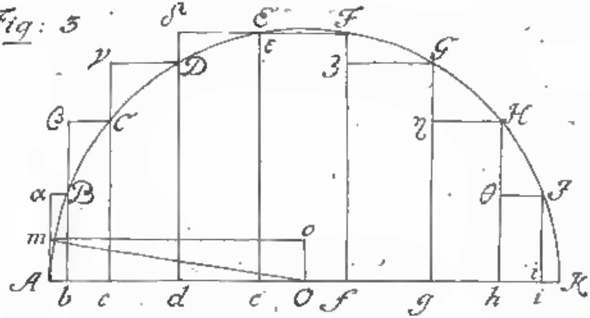


Fig: 2.

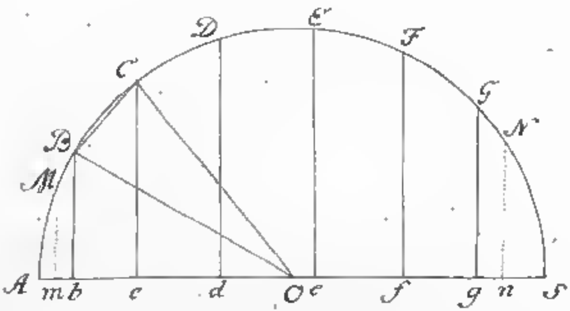
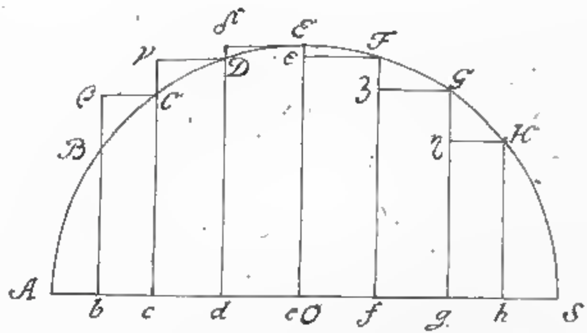


Fig 4.



F. fc.